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Exact solution of the one-impurity quantum Hall problem

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Abstract. The problem of a non-relativistic electron in the presence of a uniform electromagnetic field and of one impurity, described by means of an Aharonov–Bohm point-like vortex, is studied. The exact solution is found and the quantum Hall conductance turns out to be the same as in the impurity-free case. This exactly solvable model seems to give indications, concerning the possible microscopic mechanisms underlying the integer quantum Hall effect, which sensibly deviate from some proposals available in the literature.

1. Introduction

The discovery of the integer quantum Hall effect [1] represents one of the most remarkable experimental findings of the last few years. The effort made towards an explanation of the experimental plots for the Hall conductance versus the electron density or the applied magnetic field has stimulated huge theoretical activity†. The common key ingredients, among theoretical models that have been put forward, are the irrelevance of electron interactions and the central role played by the presence of impurities within the Hall sample, i.e. the effect of disorder. It appears therefrom, that the study of the $(2 + 1)$-dimensional quantum dynamics of a non-relativistic electron, in the presence of background electromagnetic fields and of suitable potentials describing disorder, is of essential importance. Such a simple model should represent a natural starting point in order to achieve a microscopic description of the integer quantum Hall phenomenology.

The simplest way to describe one localized impurity would appear to be a point-like interaction as *naively* described by a δ-like potential‡. However, it turns out that a quantum mechanical δ-like potential in two spatial dimensions is mathematically ill-defined [4] and one has to study the most general boundary conditions for the wavefunctions at the impurity position. In such a way, conditions of regularity of the wavefunctions mean the absence of a point-like or contact interaction, whereas non-trivial singular boundary conditions of the wavefunctions at some point entail the presence of a point-like impurity at that point. To do this, eventually, the analysis of deficiency indices and subspaces should be worked out in order to find all the self-adjoint extensions of the corresponding quantum Hamiltonian, which has

† A recent review on the subject is given in [2].

‡ In [3] it is claimed that the model with a single δ-function impurity is 'essentially exactly solvable'. However, several approximations and formal manipulations, which are not mathematically clean, are involved. A rigorous treatment of the contact interaction in quantum mechanics [4] drives us to different conclusions as we shall see in the following.

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not been done so far in the presence of uniform electric and magnetic fields. Actually, a more general and mathematically consistent way to model one point-like impurity is by means of a point-like vortex potential [5] of the Aharonov–Bohm type [6] together with all its possible self-adjoint extensions, i.e. all the possible boundary conditions at the vortex position. In so doing, taking the limit of a vanishing vortex flux, one comes back smoothly to the above case of the pure contact interaction. In contrast, the limit of vanishing electric Hall field (which is one that breaks the $O(2)$ symmetry) is very delicate, as it turns out to be entangled with the nature of the domain of the quantum Hamiltonian.

Anyway, apart from the detailed shape of the disorder potentials, the theoretical investigations concerning the calculation of the density of the quantum states, as well as of the electric conductivity, actually rely upon approximate methods and, notably, perturbative approaches [2, 5, 7, 8]. Nonetheless, it turns out (to the best of our knowledge) that no exact non-perturbative solutions have been obtained, even in the simplest realizations of the above-mentioned basic model of the two-dimensional electron ideal gas in the presence of disorder potentials.

It is the aim of the present paper to make a first step towards filling this gap, as we shall exhibit and discuss the exact solution for the quantum mechanical problem of a non-relativistic electron in the presence of a uniform electromagnetic field and of the Aharonov–Bohm vortex plus contact interaction potential to describe one impurity. In spite of its apparent simplicity, the solution of the latter model is not trivial. As a matter of fact, while in the absence of the electric field the $O(2)$ rotational symmetry of the model naturally suggests the use of polar coordinates and of the symmetric gauge, switching on of the symmetry-breaking uniform electric field does indeed spoil that possibility. Consequently, it appears to be extremely fruitful, in order to find the exact solution, to follow an algebraic method as well as to employ holomorphic coordinates [9]. Moreover, to reach our final goal it is necessary to perform a little mathematical *tour de force*, in order to become familiar with the realm of the self-adjoint extensions of the symmetric radial Hamiltonian operators, i.e. to specify the nature of their domains†.

The present analysis shows that, in the absence of the electric field, a large—actually infinite—degree of arbitrarity is allowed in the specification of the quantum radial Hamiltonian operators. In contrast, after the addition of a non-vanishing uniform electric field, the situation changes drastically: 'localized eigenstates' are no longer allowed in the one-impurity model, all the eigenstates being improper and non-degenerate—just like in the impurity-free case, cause the Hamiltonian to turn out to be essentially self-adjoint. This is the ultimate reason why the exact solution is unique. On the other hand, it is also found that the wavefunctions of the improper and non-degenerate conducting eigenstates are necessarily singular at the impurity position, but in the limiting case of a vanishing vortex flux, a clear signal that the configuration manifold underlying the model is that of the one-punctured plane, i.e. topologically non-trivial.

All those above-mentioned features of the model do represent the tools, thanks to which the total Hall conductance is computed to be the same as in the 'classical' impurity-free problem, according again to the general consensus. However, it should not be missed by the attentive reader that the detailed quantum mechanical microscopic mechanism, which eventually drives the very same current and conductance in the zero- and one-impurity models, looks to be rather different from those usually acknowledged [2, 8]. In this sense we hope that the exact solution of the one-impurity model could shed some light on the intimate microscopic nature leading on the onset of the Hall plateaux.

† Concerning definitions and basic theorems on symmetric, self-adjoint and essentially self-adjoint operators see, e.g., [13].

2. Exactly solvable models

In this section we briefly review the exact solutions for the non-relativistic quantum mechanical motion of a charged point-like particle of charge $-|e|$ and mass m (one electron) in $2+1$ dimensions, first in the presence of uniform (i.e. constant and homogeneous) electric and magnetic fields (known as the 'classical' Landau problem) and, second, in the presence of a uniform magnetic field and of one impurity described by a point-like Aharonov–Bohm vortex. Although these solutions are very well known, we find it useful to reproduce the results within the so-called symmetric gauge and paying special attention to some algebraic methods that will be quite useful later on, in order to treat the one-impurity Hall problem. In so doing, we also establish our notation and conventions.

After choosing a uniform magnetic field of strength $B > 0$ and orthogonal to the Ox_1x_2 plane in the symmetric gauge

$$A_j(x_1, x_2) = -\tfrac{1}{2}\epsilon_{jl}x_l B \qquad j, l = 1, 2 \qquad \epsilon_{12} = 1 \tag{2.1}$$

we set

$$z = \frac{x_1 + \mathrm{i}x_2}{\lambda_B} = x + \mathrm{i}y \tag{2.2a}$$

$$x_1 = \lambda_B \frac{z + \bar{z}}{2} \qquad x_2 = \lambda_B \frac{z - \bar{z}}{2\mathrm{i}} \tag{2.2b}$$

$$\partial_z = (\lambda_B/2)\,(\partial_1 - \mathrm{i}\partial_2) \tag{2.2c}$$

where $\lambda_B = \sqrt{\hbar c/|e|B}$ is the magnetic length. One can now immediately realize the Schrödinger–Pauli Hamiltonian for the spin-up component† in the presence of an additional uniform electric Hall field of strength $E_H > 0$, along the positive Ox_1 direction:

$$H(E_H) = \frac{\hbar^2}{2m\lambda_B^2}\left(2\bar{\delta}\delta - \tfrac{1}{2}\varrho(z + \bar{z})\right) \tag{2.3}$$

where the energy creation–annihilation operators $\delta \equiv \mathrm{i}\sqrt{2}\,\{\partial_{\bar{z}} + (z/4)\} = \bar{\delta}^\dagger$ and the dimensionless parameter $\varrho \equiv 2(E_H/B)\sqrt{mc^2/\hbar\omega}$ have been introduced, $\omega \equiv (|e|B/mc)$ being the classical cyclotron angular frequency. The above Hamiltonian operator, whose domain is that of the regular wavefunctions on the plane, turns out to be self-adjoint since, as we shall see, the eigenvalues are real and the eigenfunctions span a complete orthonormal set.

Now, there is a nice algebraic way to put the above Hamiltonian into diagonal form. To this end, let us first introduce the following set of translated energy and degeneracy creation–annihilation operators, respectively. Namely,

$$\delta_\varrho \equiv \mathrm{i}\sqrt{2}\left\{\partial_{\bar{z}} + \tfrac{1}{4}(z - \varrho)\right\} = \bar{\delta}_\varrho^\dagger \tag{2.4a}$$

$$\bar{\delta}_\varrho \equiv \mathrm{i}\sqrt{2}\left\{\partial_z - \tfrac{1}{4}(\bar{z} - \varrho)\right\} = \delta_\varrho^\dagger \tag{2.4b}$$

$$\theta_\varrho \equiv -\mathrm{i}\sqrt{2}\left\{\partial_z + \tfrac{1}{4}(\bar{z} - \varrho)\right\} = \bar{\theta}_\varrho^\dagger \tag{2.4c}$$

$$\bar{\theta}_\varrho \equiv -\mathrm{i}\sqrt{2}\left\{\partial_{\bar{z}} - \tfrac{1}{4}(z - \varrho)\right\} = \theta_\varrho^\dagger \tag{2.4d}$$

which fulfil the operator algebra

$$\left[\delta_\varrho, \bar{\delta}_\varrho\right] = \left[\theta_\varrho, \bar{\theta}_\varrho\right] = 1 \qquad \left[\delta_\varrho, \theta_\varrho\right] = \left[\delta_\varrho, \bar{\theta}_\varrho\right] = 0 \tag{2.5}$$

† Throughout this paper we shall always refer to the spin-up components of the wavefunctions, the generalization to the spin-down components being straightforward.

then, it is a simple exercise to show that the Hamiltonian operator (2.3)—up to the energy scale factor $(\hbar^2/2m\lambda_B^2)$—can be cast into the form

$$\frac{2m\lambda_B^2}{\hbar^2}H(E_H) \equiv \mathtt{h}(\varrho) = 2\bar{\delta}_\varrho\delta_\varrho + \mathrm{i}\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \tfrac{3}{4}\varrho^2. \tag{2.6}$$

The above expression for the Hamiltonian, together with the operator algebra (2.5), actually suggest that we can search for the eigenvectors of $\mathtt{h}(\varrho)$ as simultaneous eigenstates of the 'Landau-like' Hamiltonian $2\bar{\delta}_\varrho\delta_\varrho$ and of the operator

$$\mathtt{T}(\varrho) \equiv \mathrm{i}\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \tfrac{3}{4}\varrho^2 = -\varrho\left(\frac{\lambda_B p_2}{\hbar} + \frac{x_1 - \varrho\lambda_B}{2\lambda_B} + \tfrac{3}{4}\varrho\right) \tag{2.7}$$

which admits a continuous spectrum and represents the combined effect of a translation along the Ox_2-axis and a gauge transformation. As a matter of fact, if we introduce the real number $p_\perp \equiv (\lambda_B p_2/\hbar)$ where p_2 is the transverse momentum—orthogonal to the electric field—we obtain

$$\mathtt{T}(\varrho)\exp\left\{\mathrm{i}y\left(p_\perp - \tfrac{1}{2}x\right)\right\} = \left(-\varrho p_\perp - \tfrac{1}{4}\varrho^2\right)\exp\left\{\mathrm{i}y\left(p_\perp - \tfrac{1}{2}x\right)\right\}. \tag{2.8}$$

It follows therefore, that the Hamiltonian (2.6) has a continuous non-degenerate spectrum whose eigenvalues are given by

$$\varepsilon_{n,p_\perp} = 2n - \varrho p_\perp - \tfrac{1}{4}\varrho^2 \qquad n+1 \in \mathbb{N} \quad p_\perp \in \mathbb{R} \tag{2.9}$$

which reproduce the well known electric field splitting of the Landau bands.

The eigenfunctions can be written in the following forms, namely,

$$\begin{aligned}\langle xy|\psi_{n,p_\perp}(\varrho)\rangle \equiv \psi_{n,p_\perp}(x,y;\varrho) &= \sum_{k=0}^{\infty} c_k^{(n)}(p_\perp)\varphi_{n,k}(z,\bar{z};\varrho) \\ &= u_n\left(x - \tfrac{1}{2}\varrho - p_\perp\right)\exp\left\{\mathrm{i}yp_\perp - \tfrac{1}{2}\mathrm{i}xy\right\}\sqrt{\frac{1}{2\pi}}\end{aligned} \tag{2.10}$$

in which

$$c_k^{(n)}(\tilde{p}) = (-\mathrm{i})^k u_k(\tilde{p}) \qquad k+1, n+1 \in \mathbb{N} \tag{2.11}$$

where $\{u_k,\ k+1 \in \mathbb{N}\}$ is the complete orthonormal set of Hermite functions and

$$\begin{aligned}\langle z\bar{z}|n,k;\varrho\rangle \equiv \varphi_{n,k}(z,\bar{z};\varrho) = \sqrt{\frac{1}{2\pi}}\exp\left\{-\tfrac{1}{4}(\bar{z}-\varrho)(z-\varrho)\right\}\left(-\mathrm{i}\frac{\bar{z}-\varrho}{\sqrt{2}}\right)^n\left(\mathrm{i}\frac{z-\varrho}{\sqrt{2}}\right)^k \\ \times\sum_{h=0}^{\infty}\left(-\frac{2}{|z-\varrho|^2}\right)^h\frac{\sqrt{n!k!}}{h!\,\Gamma(n+1-h)\Gamma(k+1-h)}\end{aligned} \tag{2.12}$$

is the complete orthonormal Bargmann–Segal set. It readily follows that the improper eigenfunctions (2.10) are complete and orthonormalized in the continuum as

$$\langle\psi_{n,p_\perp}(\varrho)|\psi_{m,q_\perp}(\varrho)\rangle = \delta_{n,m}\delta(p_\perp - q_\perp) \qquad p_\perp, q_\perp \in \mathbb{R} \quad n+1, m+1 \in \mathbb{N}. \tag{2.13}$$

Finally, we briefly recall that the 'classical' Hall conductance of each eigenstate is provided by

$$\sigma_{xy} = -\frac{e^2}{h}\Gamma_L^{-1} \tag{2.14}$$

where the usual Landau levels degeneracy factor is

$$\Gamma_L \equiv \frac{1}{2\pi\lambda_B^2} = \frac{|e|B}{hc}. \tag{2.15}$$

According to the above described simple property, it turns out that the total Hall conductance of an ideal electron gas in a pure sample is proportional to the filling factor $\nu = (\mathtt{n}/\Gamma_L)$, where $\mathtt{n}$ denotes the number of electrons *per* unit area.

The problem of a point-like charged particle on the plane in the presence of a uniform magnetic field and one Aharonov–Bohm point-like singularity (the AB-vortex) has already been solved in the literature [10]. It turns out that the AB-vortex faithfully describes [5] the presence of some localized impurity within the Hall sample. It is quite instructive to resolve this problem by means of a suitable algebraic method. In so doing, in fact, it is possible to unravel some interesting features of the exact solutions, which have not yet been discussed so far, to the best of our knowledge, but will be crucial in order to provide the exact solution in the presence of an additional uniform electric field.

The gauge potential, in the symmetric gauge, is now given by

$$A_j(x_1, x_2) = -\epsilon_{jl}x_l\left(\frac{B}{2} - \frac{(\phi/2\pi)}{x_1^2 + x_2^2}\right) \tag{2.16}$$

in which the flux parameter $\phi > 0$ (< 0) means that the vortex magnetic field, located at the origin, is antiparallel (parallel) to the uniform magnetic field $B > 0$. After introduction of the quantum flux unity $\phi_0 \equiv (hc/|e|)$ and of the dimensionless parameter $\alpha \equiv (\phi/\phi_0)$, it can be easily shown that the rescaled Schrödinger–Pauli Hamiltonian for the upper spinor component takes the form

$$\mathtt{h}(\alpha) = 2\bar{\delta}(\alpha)\delta(\alpha) \tag{2.17}$$

where the singular creation–annihilation energy operators appear to be

$$\delta(\alpha) \equiv \mathrm{i}\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z}{4}\left(1 - \frac{\alpha}{[\gamma]}\right)\right\} = \bar{\delta}^{\dagger}(\alpha) \tag{2.18a}$$

$$\bar{\delta}(\alpha) \equiv \mathrm{i}\sqrt{2}\left\{\partial_{z} - \frac{\bar{z}}{4}\left(1 - \frac{\alpha}{[\gamma]}\right)\right\} = \delta^{\dagger}(\alpha) \tag{2.18b}$$

with $\gamma \equiv (\bar{z}z/2)$. The singularity at $\gamma = 0$ in the foregoing expressions is understood in the sense of the tempered distributions [11]; namely,

$$\begin{aligned}\frac{1}{[\gamma]} &\equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\ln\sqrt{x^2 + y^2}\right)^2 + C\delta(x)\delta(y) \\ &= \tfrac{1}{4}\Delta\ln^2(\bar{z}z) + C\delta^{(2)}(\bar{z}, z)\end{aligned} \tag{2.19}$$

where C is an arbitrary constant, the presence of which ensures *naive* scaling behaviour of the tempered distribution itself, i.e. $1/[c\gamma] = (1/c)(1/[\gamma]),\ c > 0$. To start with, let us consider the domain of the operators (2.18) to be the Besov's space $\mathcal{T}(\mathbb{R}^2) = \{f \in \mathcal{S}(\mathbb{R}^2)|\ f(0) = 0\}$, which is dense in $L^2(\mathbb{R}^2)$.

To be definite and without loss of generality, we shall choose in the following $-1 < \alpha < 0$, corresponding to parallel uniform and vortex magnetic fields. As a matter of fact, it is well known that only the non-integer part of α is relevant, its integer part always being reabsorbed by means of a single-valued gauge transformation.

In order to fully solve the eigenvalue problem, it is convenient to also introduce the associated singular creation–annihilation degeneracy operators

$$\theta(\alpha) \equiv -\mathrm{i}\sqrt{2}\left\{\partial_z + \frac{\bar{z}}{4}\left(1 + \frac{\alpha}{[\gamma]}\right)\right\} = \bar{\theta}^{\dagger}(\alpha) \tag{2.20a}$$

$$\bar{\theta}(\alpha) \equiv -\mathrm{i}\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z}{4}\left(1 + \frac{\alpha}{[\gamma]}\right)\right\} = \theta^{\dagger}(\alpha) \tag{2.20b}$$

always acting on the same domain $\mathcal{T}(\mathbb{R}^2)$, in such a way that the following simple operator algebra still holds true for any $-1 < \alpha \leqslant 0$; namely,

$$[\delta(\alpha), \bar{\delta}(\alpha)] = [\theta(\alpha), \bar{\theta}(\alpha)] = 1 \qquad [\delta(\alpha), \theta(\alpha)] = [\delta(\alpha), \bar{\theta}(\alpha)] = 0. \tag{2.21}$$

We note that, in order to reproduce the foregoing algebra, it is essential to employ the definition (2.19). As a matter of fact, equation (2.19) guarantees the *naive* action of the dilation operator

$$\mathtt{D}\frac{1}{[\gamma]} = -2\frac{1}{[\gamma]} \qquad \mathtt{D} \equiv z\partial_z + \bar{z}\partial_{\bar{z}} \tag{2.22}$$

whence it is an easy exercise to check the algebra (2.21).

Now, owing to the $O(2)$-symmetry of the problem, we can search for common eigenfunctions of the rescaled Hamiltonian (2.17) and of the angular momentum operator

$$\mathtt{L} \equiv \hbar(z\partial_z - \bar{z}\partial_{\bar{z}}) = \hbar\{|\alpha|\mathbf{1} + \bar{\theta}(\alpha)\theta(\alpha) - \bar{\delta}(\alpha)\delta(\alpha)\} \tag{2.23}$$

which manifestly commutes with $\mathtt{h}(\alpha)$.

Let us consider in the present section the case in which the domain of the rescaled Hamiltonian (2.17) is $\mathcal{S}(\mathbb{R}^2)$—wavefunctions regular at the origin—which is dense in $L^2(\mathbb{R}^2)$ and provides the standard solution given in the literature [10]. The eigenstates are naturally separated into two classes called integer-valued energy eigenstates (IVE), which form an infinite degenerate set, and real-valued energy eigenstates (RVE) whose degeneracy is always finite. Now, it turns out that the nth Landau band of rescaled energy $\check{\varepsilon}_n = 2n$ is spanned by the IVE eigenstates

$$\begin{aligned}\langle z\bar{z}|n < k; \check{\alpha}\rangle &= (-1)^n\sqrt{\frac{n!}{2\pi\Gamma(k+\alpha+1)}}\left(\mathrm{i}\frac{z}{\sqrt{2}}\right)^{k-n}\gamma^{-|\alpha|/2}\exp\{-\gamma/2\}L_n^{(k-n+\alpha)}(\gamma)\\ &\equiv \Psi_{n<k}(z,\bar{z}) \qquad k \geqslant n+1 \in \mathbb{N}\end{aligned} \tag{2.24}$$

$L_n^{(\beta)}$ being the generalized Laguerre polynomials. It is worthwhile to remark that the set of above eigenstates (2.24) actually realizes the infinite degeneracy of the Landau bands, the degeneracy being labelled by the quantum number $k \geqslant n+1$. Note also that the integer-valued energy bands contain an infinite number of states, although $n+1$ states less than the corresponding ordinary Landau band in the absence of the AB-vortex impurity. Finally, all the IVE actually belong to $\mathcal{T}(\mathbb{R}^2)$, since it is easy to check that their holomorphic representations do vanish at the origin, since, as is manifest from the above expression, the items $k = 0, 1, 2, \ldots, n$ are forbidden as they drive outside either the domain of the Hamiltonian ($k = n$) or even outside $L^2(\mathbb{R}^2)$ ($k = 0, 1, \ldots, n-1$).

The (RVE) eigenstates correspond to non-integer eigenvalues $\hat{\varepsilon}_n = 2(n + |\alpha|)$ of the rescaled Hamiltonian, the corresponding eigenfunctions being

$$\begin{aligned}\langle z\bar{z}|n \geqslant k; \hat{\alpha}\rangle &= (-1)^k\sqrt{\frac{k!}{2\pi\Gamma(n-\alpha+1)}}\left(-\mathrm{i}\frac{\bar{z}}{\sqrt{2}}\right)^{n-k}\gamma^{|\alpha|/2}\exp\{-\gamma/2\}L_k^{(n-k-\alpha)}(\gamma)\\ &\equiv \Psi_{n\geqslant k}(z,\bar{z}) \qquad n+1 \in \mathbb{N} \quad 0 \leqslant k \leqslant n\end{aligned} \tag{2.25}$$

which, again, belong to the above specified domain of $\mathtt{h}(\alpha)$ iff the degeneracy quantum number k does not exceed the energy quantum number n.

3. Self-adjoint extensions of the Hamiltonian

So far we have considered the rescaled Hamiltonian $\mathtt{h}(\alpha)$ to be defined on the domain of the regular square-integrable wavefunctions. However, as is well known, this is not the most general case. Let us now consider, therefore, different quantum Hamiltonians, corresponding to different self-adjoint Hamiltonian operators, whose differential operator is always given by equation (2.17), but whose domain is now allowed to contain wavefunctions with square-integrable singularities at the origin, i.e. at the AB-vortex position. This procedure is the mathematically correct way to introduce in this context a contact-interaction or point-like interaction. In physical terms, it is equivalent to *naively* adding some kind of δ-like potential to the classical Hamiltonian. We have to stress, in fact, that, strictly speaking, δ-like potential is ill-defined in two and three spatial dimensions [4, 12] and the proper way to encompass the possibility of contact-interaction is by means of the analysis of the self-adjoint extensions of the quantum Hamiltonian. In particular, the solution we have discussed in the previous section, i.e. the case of the Hamiltonian whose domain is that of the regular wavefunctions, can be thought of as the pure AB interaction in the absence of contact-interaction. The presence of a particular square-integrable singularity of the wavefunction at the vortex position will select some new quantum Hamiltonian, which will describe the presence of a specific contact-interaction. What we shall see in the following is that there are an infinite number of such Hamiltonians, which are perfectly legitimate and turn out to describe different physics, as they are characterized by different spectra and degeneracies. As a matter of fact, it is not difficult to prove the following lemma.

Lemma (–). *In any subspace of fixed negative or vanishing angular momentum* $\ell = -\hbar l$, $l+1 \in \mathbb{N}$ *there are two options in order to specify the quantum radial Hamiltonian: if the domain is that of regular wavefunctions we have*

$$\mathtt{h}_l(\alpha) = \sum_{n=l}^{\infty} 2(n+|\alpha|)\,\hat{P}_{n>n-l}(\alpha) \tag{3.1}$$

where the projectors onto the regular RVE eigenstates of equation (2.25) are introduced, i.e. $\hat{P}_{n\geqslant k}(\alpha) \equiv \left|n \geqslant k;\hat{\alpha}\right\rangle\langle n \geqslant k;\hat{\alpha}|$. *Alternatively, if the domain is that of the wavefunctions which are square integrable on the plane, although singular at the impurity's position, we have*

$$\mathtt{H}_l(\alpha) = \sum_{n=l}^{\infty} 2n\,\check{P}_{n\geqslant n-l}(\alpha) \tag{3.2}$$

where

$$\check{P}_{n\geqslant n-l}(\alpha) \equiv |n \geqslant n-l;\check{\alpha}\rangle\langle n \geqslant n-l;\check{\alpha}| \tag{3.3}$$

the state $|n \geqslant n-l;\check{\alpha}\rangle$ *being given by*

$$\langle z\bar{z}|n \geqslant n-l;\check{\alpha}\rangle \equiv \Phi_{n>n-l}(z,\bar{z}) = \sum_{j=0}^{\infty} \check{c}_{n,l}^{\;j}\,\hat{\psi}_{j,j-l}(z,\bar{z};\alpha) \qquad l = 0,1,\ldots,n \tag{3.4}$$

where

$$\begin{aligned}\check{c}_{n,l}^{\;j} &\equiv \langle j > j-l;\hat{\alpha}|n > n-l;\check{\alpha}\rangle \\ &= \binom{j-\alpha}{n}\binom{n+\alpha-l}{j-l}\sqrt{\frac{(j-l)!n!}{\Gamma(j-\alpha+1)\Gamma(n-l+\alpha+1)}}.\end{aligned} \tag{3.5}$$

The above equation (3.5) uniquely defines a state vector $\forall n+1 \in \mathbb{N}$*, owing to the Riesz–Fisher theorem, since it can actually be verified that*

$$\sum_{k=0}^{\infty} \left|\check{c}_{n,l}^{\,k}\right|^2 = \frac{\{(n+\alpha)(\alpha)_n\}^2}{n!\,\Gamma(1-\alpha)\Gamma(n-l+\alpha+1)[(-n-\alpha)_l]^2} \sum_{k=0}^{\infty} \frac{(1-\alpha)_k}{(k-l)!(n+\alpha-k)^2} = 1. \tag{3.6}$$

Note that the wavefunctions (3.4) belong to $L^2(\mathbb{R}^2)$ *by construction, they are singular at the AB-vortex position and are eigenfunctions of the symmetric operator (2.17) with integer eigenvalues* $\check{\varepsilon}_n = 2n$ *and of the angular momentum operator (2.23) with a negative or vanishing eigenvalue* $\ell = -\hbar l$*. The net result of this construction is that, by relaxing the condition of the regularity of the wavefunctions at the AB-vortex position—which specifies a particular domain of the quantum Hamiltonian—it is possible to set up different quantum Hamiltonians, with different spectra and degeneracies, after shifting an infinite set of states (actually orthonormal and complete in the subspaces of fixed angular momenta) from the RVE sector of the eigenstates to the IVE one.*

A quite similar construction can be done for the quantum radial Hamiltonians corresponding to positive angular momenta $\ell = \hbar l$, $l \in \mathbb{N}$. Again, the result can be summarized within the following lemma.

Lemma (+). *In any subspace of fixed positive angular momentum* $\ell = \hbar l$, $l \in \mathbb{N}$ *there are two options in order to specify the quantum radial Hamiltonian: if the domain is that of regular wavefunctions we have*

$$\mathrm{h}_l(\alpha) = \sum_{n=l}^{\infty} 2n\,\check{P}_{n<n+l}(\alpha) \tag{3.7}$$

in which the projectors onto the regular IVE eigenstates of equation (2.24) are introduced. Alternatively, if the domain is that of the wavefunctions which are square integrable on the plane, although singular at the impurity's position, we have

$$\mathrm{H}_l(\alpha) = \sum_{n=l}^{\infty} 2(n+|\alpha|)\,\hat{P}_{n<n+l}(\alpha) \tag{3.8}$$

where $\hat{P}_{n<n+l}(\alpha) \equiv |n<n+l;\hat{\alpha}\rangle\langle n<n+l;\hat{\alpha}|$*, the singular state* $|n<n+l;\hat{\alpha}\rangle$ *being given by*

$$\langle z\bar{z}|n<n+l;\hat{\alpha}\rangle \equiv \Phi_{n<n+l}(z,\bar{z}) = \sum_{j=0}^{\infty} \hat{c}_{n,l}^{\,j}\,\check{\psi}_{j,j+l}(z,\bar{z};\alpha) \qquad l \in \mathbb{N} \tag{3.9}$$

where

$$\begin{aligned}\hat{c}_{n,l}^{\,j} &\equiv \langle j<j+l;\check{\alpha}|n<n+l;\hat{\alpha}\rangle \\ &= \binom{j+\alpha+l}{n+l}\binom{n-\alpha}{j}\sqrt{\frac{j!(n+l)!}{\Gamma(j+l+\alpha+1)\Gamma(n-\alpha+1)}}.\end{aligned} \tag{3.10}$$

Again, it can be readily verified that

$$\sum_{j=0}^{\infty} \left|\hat{c}_{n,l}^{\,j}\right|^2 = \frac{(n-\alpha)^2\,\{(-\alpha)_n(1+\alpha)_l\}^2}{(n+l)!\Gamma(n-\alpha+1)\Gamma(\alpha+l+1)} \sum_{k=0}^{\infty} \frac{(\alpha+l+1)_k}{k!(n-\alpha-k)^2} = 1 \tag{3.11}$$

which means that, according to the Riesz–Fisher theorem, the expansion (3.9) uniquely defines a state vector in the Hilbert space.

It is crucial to gather that the action of the lowering and raising energy and degeneracy operators, which were originally defined on $\mathcal{T}(\mathbb{R}^2)$, can be extended on the singular states in terms of their L^2-expansions, e.g.

$$\theta(\alpha)|n \geqslant n-l;\check{\alpha}\rangle \equiv \sum_{k=0}^{\infty} \check{c}_{n,l}^{\,k}\,\theta(\alpha)|k \geqslant k-l;\hat{\alpha}\rangle$$

$$= \sum_{k=1}^{\infty} \check{c}_{n,l}^{\,k}\sqrt{k-l}\,|k > k-l-1;\hat{\alpha}\rangle = \sqrt{n-l+\alpha}\,|n > n-l-1;\check{\alpha}\rangle \qquad (3.12)$$

and analogous ones for the remaining raising and lowering operators.

Concerning self-adjoint extensions, we should go a little bit further in order to reach the most general statement. Actually, it can be proved that, for any fixed value $\ell = \hbar l,\ l \in \mathbb{Z}$ of the angular momentum, there is a continuous family of self-adjoint extensions of the symmetric radial Hamiltonians which interpolates between the regular one $h_l(\alpha)$ and the singular one $H_l(\alpha)$. However, since none of those further possible self-adjoint extensions will be relevant in searching for an exact solution of the one-impurity problem in the presence of the uniform electric field, we shall no longer discuss here that quite interesting matter, but leave it to a forthcoming analysis. To sum up, we can say that the solution we have discussed in the previous section—in which the domain of the quantum Hamiltonian is that of the regular wavefunctions—actually corresponds to the one-impurity problem described by a pure AB interaction. The further possible choices of the quantum self-adjoint Hamiltonians—such that the domains contain singular wavefunctions at the vortex position—do physically represent the simultaneous presence of the AB and contact-interactions.

4. Exact solution for the one-impurity quantum Hall problem

We are now ready to discuss the exact solution in the presence of the AB-vortex—the one-impurity problem—and of a uniform electromagnetic field. According to the conventional picture [2] it is conjectured that the presence of a not too large number of localized impurities within the Hall sample is actually what is needed to account for the onset of the Hall plateaux. In this respect, it is plausibly believed that the structure in terms of Landau bands is basically kept, even in the presence of a small number of localized impurities, although the density of states among and within the Landau sub-bands is significantly changed by the presence of impurities (Landau sub-bands are broadened and depopulated). It is commonly accepted that the above pattern eventually supports, in terms of various analytical approximate methods of investigations [2], some reasonable explanation for the Hall plateaux. The basic idea behind this picture is that the switching on of a weak electric field is a smooth perturbation whose net effect is, on the one hand, to lift the degeneracy of the conducting depopulated Landau bands, whereas, on the other hand, to allow the presence of non-conducting localized eigenstates, i.e. bound states.

In contrast, as we shall see below, the exact solution of the present model shows that the switching on of the uniform Hall field E_H drastically modifies the distribution and the nature of the energy eigenstates, with respect to the situation in the absence of E_H and no matter how weak the Hall field is. In particular, all the energy eigenstates are improper, each of them does contribute to the Hall current and, moreover, the improper (i.e. extended) wavefunctions of some of the eigenstates necessarily become singular at the impurity position.

In order to prove the above statements, we shall solve our problem following a constructive approach, which makes use of all the detailed explicit information we have learned in the previous sections. As a matter of fact, what we have seen before is that the presence of the one-impurity AB-vortex might indeed realize what is widely believed: the integer-valued Landau levels are kept and, in general, further non-integer-valued energy levels do actually appear, in such a way that the IVE eigenstates degeneracy of the Landau band is lowered, albeit still infinite. This means, in turn, that the density of the states is actually sensibly modified by the presence of the impurity. Consequently, which is reasonable to expect (following the afore-mentioned popular belief), after switching on some weak uniform electric Hall field, the conductance of the remaining charged states within the Landau bands is slightly increased—with respect to the impurity-free case—in such a way that the net result for the Hall conductance is again the 'classical' one of equation (2.14). As we will see below, it turns out that the exact solution actually suggests a quite different picture.

It is not difficult to verify that the rescaled Hamiltonian differential operator, in the presence of an additional uniform electric field suitably described by the afore-introduced parameter ϱ (see equation (2.3)) can be written in the form

$$\frac{2m\lambda_B^2}{\hbar^2}H(\alpha, E_H) \equiv \mathtt{h}(\alpha, \varrho) = 2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha) + \mathrm{i}\frac{\varrho}{\sqrt{2}}[\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)] - \tfrac{3}{4}\varrho^2 \tag{4.1}$$

in which the translated energy and degeneracy creation–annihilation operators are, respectively,

$$\delta_\varrho(\alpha) \equiv \mathrm{i}\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z}{4}\left(1 - \frac{\alpha}{[\gamma]}\right) - \frac{\varrho}{4}\right\} = \bar{\delta}_\varrho^\dagger(\alpha) \tag{4.2a}$$

$$\bar{\delta}_\varrho(\alpha) \equiv \mathrm{i}\sqrt{2}\left\{\partial_z - \frac{\bar{z}}{4}\left(1 - \frac{\alpha}{[\gamma]}\right) + \frac{\varrho}{4}\right\} = \delta_\varrho^\dagger(\alpha) \tag{4.2b}$$

$$\theta_\varrho(\alpha) \equiv -\mathrm{i}\sqrt{2}\left\{\partial_z + \frac{\bar{z}}{4}\left(1 + \frac{\alpha}{[\gamma]}\right) - \frac{\varrho}{4}\right\} = \bar{\theta}_\varrho^\dagger(\alpha) \tag{4.2c}$$

$$\bar{\theta}_\varrho(\alpha) \equiv -\mathrm{i}\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z}{4}\left(1 + \frac{\alpha}{[\gamma]}\right) + \frac{\varrho}{4}\right\} = \theta_\varrho^\dagger(\alpha). \tag{4.2d}$$

Again, the following canonical commutation relations hold true; namely,

$$[\delta_\varrho(\alpha), \bar{\delta}_\varrho(\alpha)] = [\theta_\varrho(\alpha), \bar{\theta}_\varrho(\alpha)] = 1 \qquad [\delta_\varrho(\alpha), \theta_\varrho(\alpha)] = [\delta_\varrho(\alpha), \bar{\theta}_\varrho(\alpha)] = 0. \tag{4.3}$$

Now, owing to the above algebra, we have that the full Hamiltonian differential operator $\mathtt{h}(\alpha, \varrho)$ and the translated 'Landau-like' differential operator $2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$ do indeed commute, i.e.

$$\left[\mathtt{h}(\alpha, \varrho), \bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)\right] = 0. \tag{4.4}$$

It is important to gather that, unless we specify the domains of the above-mentioned Hamiltonian differential operators, they are only symmetric. Since we have to deal with well defined self-adjoint quantum Hamiltonians, we have to specify the (common) domain in which the commutation relation (4.4) still holds for the corresponding quantum self-adjoint Hamiltonians. However, then, the fundamental theorem states that the self-adjoint realizations of the full rescaled Hamiltonian and of the translated 'Landau-like' Hamiltonian must have a complete orthonormal set of common eigenstates.

First, we prove that there is only one choice of the domain of the quantum Hamiltonians which allows for a solution of the problem. As a matter of fact, it appears that the spectrum of

any self-adjoint extension $\mathtt{H}(\alpha, \varrho)$ of the full rescaled Hamiltonian (4.1) is continuous owing to the presence of the degeneracy lifting operator

$$\mathtt{T}(\alpha, \varrho) \equiv \mathrm{i}\frac{\varrho}{\sqrt{2}}[\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)] - \tfrac{3}{4}\varrho^2 \tag{4.5}$$

whose spectrum is manifestly continuous—in the 'classical' impurity-free case it drives the electric splitting of the Landau degeneracy, see equation (2.9). Consequently, the eigenfunctions of $\mathtt{H}(\alpha, \varrho)$ will be improper state vectors and must be, owing to $[\mathtt{H}(\alpha, \varrho), \Delta_L(\alpha, \varrho)] = 0$, common eigenstates of the corresponding self-adjoint extension $\Delta_L(\alpha, \varrho)$ and of the translated 'Landau-like' Hamiltonian $2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$, whose spectrum is instead purely discrete. Consequently, the discrete energy levels of $\Delta_L(\alpha, \varrho)$ with a finite degeneracy are forbidden, because the degenerate states are proper state vectors and a finite combination of them cannot produce an improper state. This means, in particular, that if we choose the domain to be, for example, that of the regular wavefunctions on the plane, then the full rescaled Hamiltonian (4.1) is not a self-adjoint operator.

This quite general and rigorous result is such a stringent constraint that we are left with only two possible options in order to obtain a solution, namely we have to investigate the two self-adjoint extensions of the translated 'Landau-like' Hamiltonian whose spectra are given by either non-integer rescaled eigenvalues $\hat{\varepsilon}_n = 2(n + |\alpha|),\ n + 1 \in \mathbb{N}$, or, alternatively, by integer rescaled eigenvalues $\check{\varepsilon}_n = 2n,\ n + 1 \in \mathbb{N}$.

In the former case, the self-adjoint translated 'Landau-like' Hamiltonian is given by its spectral decomposition; namely,

$$\hat{\Delta}(\alpha, \varrho) \equiv \sum_{n=0}^{\infty} 2(n + |\alpha|) \left\{ \sum_{k=0}^{n} \hat{P}_{n\geqslant k}(\alpha, \varrho) + \sum_{k=n+1}^{\infty} \hat{P}_{n<k}(\alpha, \varrho) \right\} \tag{4.6}$$

where the projectors onto translated regular and singular states are given by, respectively,

$$\hat{P}_{n\geqslant k}(\alpha, \varrho) \equiv \left|n \geqslant k; \hat{\alpha}, \varrho\right\rangle\left\langle n \geqslant k; \hat{\alpha}, \varrho\right| \tag{4.7a}$$

$$\hat{P}_{n<k}(\alpha, \varrho) \equiv \left|n < k; \hat{\alpha}, \varrho\right\rangle\left\langle n < k; \hat{\alpha}, \varrho\right|. \tag{4.7b}$$

The explicit form of the above eigenstates, normalized to unity, is provided according to the general construction described in the previous section (see *lemmas* ($\pm$)), i.e.

$$|n, k; \hat{\alpha}, \varrho\rangle = \sqrt{\frac{\Gamma(1-\alpha)}{(k!)\Gamma(n+1-\alpha)}}[\bar{\delta}_\varrho(\alpha)]^n[\bar{\theta}_\varrho(\alpha)]^k|0, 0; \hat{\alpha}, \varrho\rangle \qquad n+1, k+1 \in \mathbb{N} \tag{4.8}$$

the holomorphic representation of the cyclic ground state being

$$\langle z\bar{z}|0, 0; \hat{\alpha}, \varrho\rangle = \frac{\gamma^{|\alpha|/2}\exp\{-\frac{1}{4}(z-\varrho)(\bar{z}-\varrho)\}}{\sqrt{2\pi\Gamma(1-\alpha)}}\frac{\exp\{-\varrho^2/4\}}{\sqrt{{}_1F_1(1-\alpha, 1; \varrho^2/2)}}. \tag{4.9}$$

Note that, among the eigenstates (4.8), the regular ones correspond to negative or vanishing angular momenta ($n \geqslant k$), whilst the singular ones correspond to positive angular momenta ($n < k$). Furthermore, it is manifest from the spectral decomposition (4.6) that the quantum number k labels the infinite discrete Landau degeneracy. It can be readily verified, taking the construction leading to equation (3.12) suitably into account, that the following properties hold true; namely,

$$\theta_\varrho(\alpha)|n, k; \hat{\alpha}, \varrho\rangle = \sqrt{k}|n, k-1; \hat{\alpha}, \varrho\rangle \tag{4.10a}$$

$$\bar{\theta}_\varrho(\alpha)|n, k; \hat{\alpha}, \varrho\rangle = \sqrt{k+1}|n, k+1; \hat{\alpha}, \varrho\rangle. \tag{4.10b}$$

Now, in order to find the solution of the eigenvalue problem for the quantum self-adjoint Hamiltonian

$$\hat{H}(\alpha, \varrho) = \hat{\Delta}(\alpha, \varrho) + \mathtt{T}(\alpha, \varrho) \tag{4.11}$$

let us consider the states

$$\begin{aligned} &|n, p_\perp; \hat{\alpha}, \varrho\rangle \equiv \sum_{k=0}^{\infty} c_k^{(n)}(\tilde{p})|n, k; \hat{\alpha}, \varrho\rangle \\ &\tilde{p} \equiv p_\perp - \tfrac{1}{2}\varrho \qquad p_\perp \in \mathbb{R} \end{aligned} \tag{4.12}$$

which are built up in close analogy with the 'classical' solution (2.10) (with $c_k^{(n)}(\tilde{p})$ given by equation (2.11)) and belong by definition to the continuous spectrum. Note that, by construction, the above states are obviously eigenstates of the self-adjoint operator (4.6). Actually, it is not difficult to verify that

$$\hat{H}(\alpha, \varrho)|n, p_\perp; \hat{\alpha}, \varrho\rangle = \left(2n - 2\alpha - \varrho p_\perp - \tfrac{1}{4}\varrho^2\right)|n, p_\perp; \hat{\alpha}, \varrho\rangle. \tag{4.13}$$

It is worthwhile to remark that the key point in obtaining the above result is the fact that the set of states (4.8) is closed with respect to the free action of the translated degeneracy operators (see equations (4.10)). This crucial feature is peculiar to the set (4.8) and, in particular, does not stay true for the other complete orthonormal set $|n, k; \check{\alpha}, \varrho\rangle$, which characterizes the self-adjoint extension of the Hamiltonian with only integer eigenvalues (such as in the 'classical' case). This is why the quantum Hamiltonian (4.11) is the only (essentially) self-adjoint operator with a continuous non-degenerate spectrum and which commutes with the 'Landau-like' self-adjoint operator (4.6), i.e. the unique solution of our problem.

Now, it can be readily checked that the conductance does not change with respect to the 'classical' impurity-free case [7]. As a matter of fact, starting again from the definition of the current operator

$$\hat{J}_{1,2} = \frac{|e|\hbar}{m\lambda_B}\hat{P}_{1,2}(\alpha) \tag{4.14}$$

in which

$$\hat{P}_1(\alpha) = -\frac{1}{\sqrt{2}}\left[\delta_\varrho(\alpha) + \bar{\delta}_\varrho(\alpha)\right] \tag{4.15a}$$

$$\hat{P}_2(\alpha) = \frac{\mathrm{i}}{\sqrt{2}}\left[\delta_\varrho(\alpha) - \bar{\delta}_\varrho(\alpha) + \frac{\mathrm{i}}{\sqrt{2}}\varrho\right] \tag{4.15b}$$

it immediately follows that, for any normalizable wavepacket

$$|n, [f]; \hat{\alpha}, \varrho\rangle = \int_{-\infty}^{+\infty} \mathrm{d}p_\perp\ f(p_\perp)|n, p_\perp; \hat{\alpha}, \varrho\rangle \qquad \int_{-\infty}^{+\infty} \mathrm{d}p_\perp\, |f(p_\perp)|^2 = 1 \tag{4.16}$$

which describes one electron in the nth conducting band, we obtain once again that the average current carried by such a state is

$$\langle n, [f]; \hat{\alpha}, \varrho|\hat{J}_1|n, [f]; \hat{\alpha}, \varrho\rangle = 0 \tag{4.17a}$$

$$\langle n, [f]; \hat{\alpha}, \varrho|\hat{J}_2|n, [f]; \hat{\alpha}, \varrho\rangle = -|e|c\frac{E_H}{B} \tag{4.17b}$$

which shows that the Hall conductance is always the 'classical' one as in equation (2.14), even in the presence of the AB-vortex. A further important remark is now in order. Taking the limit, when α goes to zero, of the self-adjoint Hamiltonian (4.11) we recover the standard

impurity-free self-adjoint Hamiltonian (2.6), whose domain is that of regular wavefunctions. This means that, if we eliminate the Aharonov–Bohm vortex potential, contact-interaction is no longer allowed in the presence of a crossed uniform electric and magnetic fields. Consequently, the analysis of the self-adjoint extensions of the quantum Hamiltonian does contradict the claim in [3] since no localized bound states are allowed. It should also be remarked that the switching on of a weak Hall electric field does represent a small and smooth perturbation on the system, iff the starting unperturbed Hamiltonian is

$$\hat{\Delta}(\alpha) \equiv \sum_{n=0}^{\infty} 2(n + |\alpha|) \left\{ \sum_{k=0}^{n} \hat{P}_{n \geqslant k}(\alpha) + \sum_{k=n+1}^{\infty} \hat{P}_{n<k}(\alpha) \right\}. \tag{4.18}$$

Otherwise, for any different choice of the unperturbed Hamiltonian, the additional electric field cannot represent a smooth perturbation, since it involves a change in the domain of the Hamiltonian.

5. Conclusion

In this paper we have explicitly solved the quantum mechanical $(2 + 1)$-dimensional problem of the non-relativistic electron in the presence of a uniform electromagnetic field and of an Aharonov–Bohm vortex potential. The solution is unique, since it turns out that the quantum Hamiltonian is essentially self-adjoint in the presence of the uniform electric field. This is no longer true in the absence of the electric field: in the latter case (under the assumption of $O(2)$-symmetry) each radial Hamiltonian allows for a one-parameter family of self-adjoint extensions. From explicit knowledge of the eigenvalues and eigenfunctions of the full Hamiltonian, it is possible to compute the current and conductance, the results being the same as in the 'classical' case, i.e. in the absence of the AB-vortex. It should be emphasized that the possibility to obtain the exact non-perturbative solution is heavily supported by the systematic application of the algebraic method, which allows us to overcome the conflict between the rotational symmetry in the absence of the electric field and the explicit symmetry breaking due to the switching on of the uniform electric field itself.

The final result is that the Hall conductance does not change in the presence of one impurity described by the AB-vortex. The microscopic picture which emerges from the exact solution of the present one-impurity model can be summarized as follows. In the absence of the electric field, the general pattern can be described, as was basically known [10], by the presence of a depopulated integer-valued Landau levels and of further real-valued energy levels of finite degeneracy, with the details of this description depending upon the specific self-adjoint extension of the quantum Hamiltonian, as was carefully explained in section 3.

On the grounds of this model, one is led to the picture of the 'broadening' of the Landau sub-bands, owing to the presence of impurities, and to the idea that the switching on of a weak electric field does basically keep this feature: the Hall conducting states of the electrons should be only those ones lying within the integer-valued Landau sub-bands, the remaining allowed bound states giving no contribution to the Hall conductivity. In contrast, the exact solution of the present simple model shows that the switching on of the uniform electric field drastically changes the above picture: the electrically split Landau levels are shifted with respect to the impurity-free case—see equations (2.9) and (4.13)—all the energy eigenstates belong to the electrically split Landau sub-bands and the quantum eigenstates within each sub-band are necessarily described by a singular wavefunction, at variance with the impurity-free case. It is quite remarkable that, in spite of the above drastic reshuffling of the quantum states after the switching on of the electric field, the current and conductance are exactly the same with

and without the AB-vortex. This fact appears to corroborate some deep topological nature of the Hall conductance, as was widely recognized in the literature [2]. As a matter of fact, the unavoidable presence of singularities of the eigenfunctions at the vortex position clearly represents the presence of a topological non-triviality. In other words, since the exact solution of the one-impurity problem necessarily involves singular wavefunctions at the impurity position, it means that the underlying configuration manifold in the general dynamical problem is the punctured plane, which is topologically non-trivial. This is *a fortiori* true in the realistic many-impurities problem, whose exact solution is still unknown.

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